

On the Density of Identifying Codes in the Square Lattice

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$B_r(v) \cap C$, $v \in V$, are all nonempty and different, where $B_r(v)$ denotes the set of all points within distance r from v , we call C an r -identifying code. We give bounds on the best possible density of r -identifying codes in the two-dimensional square lattice. © 2002 Elsevier Science (USA)

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1. INTRODUCTION

Given an undirected graph $G = (V, E)$, define $B_r(v)$, the *ball* of radius r centred at a vertex $v \in V$, by

$$B_r(v) = \{x \in V : d(x, v) \leq r\},$$

where $d(x, v)$ represents the number of edges in any shortest path between v and x . We call any nonempty subset C of V a *code* and its elements *codewords*. A code C is called an r -identifying code if the sets $B_r(v) \cap C$, $v \in V$, are all nonempty and different. This concept was introduced by Karpovsky *et al.* [11]. The motivation comes from fault diagnosis in arrays of processors. A multiprocessor system can be modeled as an undirected graph $G = (V, E)$ where V is the set of processors and E is the set of links between the processors. Assume that at most one of the processors is malfunctioning and we wish to test the system and locate the faulty processor. For this purpose a set of processors will be selected and

they will be assigned the task of testing themselves and their neighbourhoods. Whenever a selected processor detects a fault of any kind either in itself or in any processor within distance r , it sends an alarm signal. We require that we can uniquely tell the location of the malfunctioning processor based only on the information which ones of the selected processors sent the alarm signal.

In this paper, we take as our graph the two-dimensional square lattice T with the vertex set $V = \mathbb{Z}^2$ and edge set $E = \{\{u, v\} : u - v = (\pm 1, 0) \text{ or } u - v = (0, \pm 1)\}$.

Denote by Q_n the set of vertices $(x, y) \in V$ with $|x| \leq n$ and $|y| \leq n$. Then the density of C is defined as

$$D(C) = \limsup_{n \rightarrow \infty} \frac{|C \cap Q_n|}{|Q_n|}.$$

In this paper, we investigate the problem of how small the density of an r -identifying code $C \subseteq \mathbb{Z}^2$ can be. We prove that, for all r , the density of an r -identifying code is at least $2/(7r+4)$ and for all even r (resp. odd r) we construct an r -identifying code with density $2/(5r)$ (resp. $2r/(5r^2 - 2r + 1)$). The previous best general lower bound was $1/(4r+2)$ [7]. For $r=1$, the lower and upper bounds $15/43$ and $7/20$ were established in [7] and [3], respectively. For $r=2$, we give here a construction with density $5/29$.

A code $C \subseteq V$ is called a *locating-dominating set* if the sets $B_1(v) \cap C$ are nonempty and different for $v \in V \setminus C$, i.e., only for the non-codewords v . For this closely related problem, we refer to Slater [14], Rall and Slater [13], and the book of Haynes *et al.* [9]. It has been shown by Slater [15] that $3/10$ is the smallest possible density of a locating-dominating set in the square lattice.

2. A GENERAL LOWER BOUND

THEOREM 2.1. *If C is an r -identifying code in the square lattice, then its density D satisfies the inequality*

$$D \geq \frac{2}{7r+4}.$$

Proof. Let $r \geq 1$ be fixed.

For every edge $\{u, v\}$ in the square lattice, there has to be a codeword c that is within distance r from exactly one of the points u and v . Suppose that a codeword c is given: For which edges does it do the job?

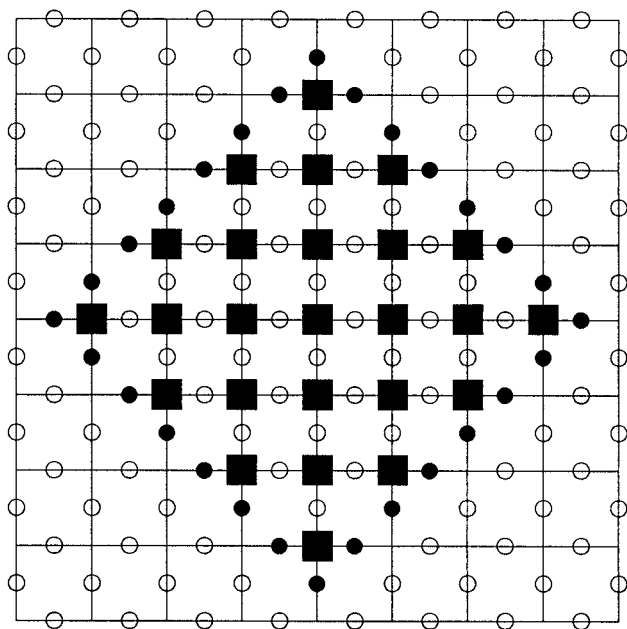


FIG. 1. The black squares form a ball of radius three; the solid circles mark the edges exactly one of whose endpoints is in the ball.

The situation is illustrated in Fig. 1 for $r = 3$. The edges with the property that c is within distance r from exactly one of the endpoints are marked by solid circles. In general, we get a $(2r+2) \times (2r+2)$ square consisting of $4(2r+1)$ solid circles.

If we now represent each edge by its middle point, then these middle points form another square lattice, in different scale from the original one and rotated by 45 degrees; see Fig. 2.

Consequently, we must cover the new square lattice with $(2r+2) \times (2r+2)$ squares.

Let now K be a large integer and consider the set

$$A = \{(x, y) \in \mathbb{R}^2: |x| \leq K, |y| \leq K\}$$

containing $(2K+1)^2$ points of the original lattice and $4K(2K+1)$ points of our new lattice. Assume that we have a set of squares that together cover all the circles, i.e., the points in the new lattice. Consider all the squares whose centres are in A : let these centre points be c_1, c_2, \dots, c_M and denote the corresponding squares by S_1, S_2, \dots, S_M .

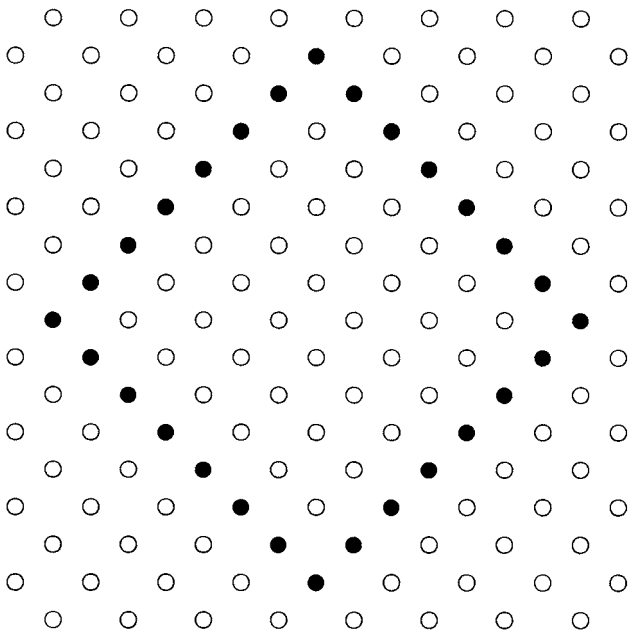


FIG. 2. The new square lattice formed by the midpoints of the edges in the old square lattice.

Denote

$$e_i = \left| S_i \setminus \bigcup_{j < i} S_j \right|$$

and

$$f_i = \left| S_i \setminus \bigcup_{j > i} S_j \right|.$$

Then

$$\sum_{i=1}^M e_i \geq 4K(2K+1) - \alpha K \tag{1}$$

and

$$\sum_{i=1}^M f_i \geq 4K(2K+1) - \alpha K, \tag{2}$$

where α is a suitable positive constant which depends on r but not on K . Indeed, to get the first inequality we just go through the codewords c_1, c_2, \dots, c_M , and to get the second inequality we go through these codewords in the reverse order c_M, c_{M-1}, \dots, c_1 . In both cases we have to subtract a small term, which is linear in K , because some of the circles in A which lie close to the border may be covered by squares whose centres do not lie in A .

Assume that

$$c_k \in \{(x, y) \in \mathbb{R}^2: |x| \leq K - 2r, |y| \leq K - 2r\}. \quad (3)$$

In particular, all the points in S_k are in A .

Let us assume that S_k is as in Fig. 2. Denote the left-most black circle by $v(0, 0)$, the lowest one by $v(2r+1, 0)$, the right-most by $v(2r+1, 2r+1)$ and the remaining corner by $v(0, 2r+1)$, and extend the numbering to the other points in the natural way. Each of the points $v(1, 1), v(2, 2), \dots, v(2r, 2r)$ must also be covered by a square. By (3), the centres of all these squares lie in A . All the squares have the same size, and this implies that the square that covers $v(i, i)$ ($1 \leq i \leq 2r$) must cover at least one of the points $v(0, i), v(i, 0), v(2r+1, i)$ and $v(i, 2r+1)$. Therefore there are at least $2r$ points in the square S_k that are covered by more than one square, and each of them is covered by at least one S_j with $j < k$ or $j > k$ (maybe both). Hence

$$e_k + f_k \leq 4(2r+1) + 4(2r+1) - 2r = 14r + 8.$$

There are fewer than $16rK$ codewords among c_1, c_2, \dots, c_M that do not satisfy (3). For them, $e_k + f_k \leq 16r + 8$. Together with (1) and (2) we get

$$(14r + 8)M + 16rK \cdot 2r \geq 8K(2K + 1) - 2\alpha K.$$

Hence the proportion of codewords within A satisfies

$$\frac{M}{(2K+1)^2} \geq \frac{8K(2K+1) - 2\alpha K - 32r^2K}{(14r+8)(2K+1)^2}.$$

Letting now K tend to infinity we get the lower bound $D \geq 2/(7r+4)$ as claimed. ■

3. GENERAL CONSTRUCTIONS

It is convenient to denote

$$I(v) = B_r(v) \cap C$$

for $v \in V$.

THEOREM 3.1. *For every even r , there is an r -identifying code with density $2/(5r)$ in the square lattice.*

Proof. Let

$$C(0, 0) = \{(0, 0), (2, 0), (4, 0), \dots, (2r - 2, 0)\}$$

and

$$C(i, j) = C(0, 0) + i(5r/2, r/2) + j(0, r).$$

We claim that C , the union of all $C(i, j)$, $i, j \in \mathbb{Z}$, is an r -identifying code. The case $r = 6$ is shown in Fig. 3. The codewords have been marked with solid dots.

The lattice points v for which $I(v)$ consists of only points in $C(0, 0)$ form the set $L(0, 0) = \{(-1, 0), (1, 0), (3, 0), \dots, (2r - 1, 0)\}$ and if this is the case we can immediately identify v . The set $L(i, j)$ is defined analogously.

We can also immediately tell, if $v \in C(i, j)$ for some i, j : this is the only case when $I(v)$ contains at least one point from each of the sets $C(i, j - 1)$, $C(i, j)$ and $C(i, j + 1)$.

Denote by $H(i, j)$ the set of all lattice points v other than the ones in $C(i, j)$ and $C(i, j + 1)$ such that $I(v)$ contains at least one point from both $C(i, j)$ and $C(i, j + 1)$. The set of such points has a hexagonal shape (cf. Fig. 3), the borders are included, except the lattice points on the two longest sides.

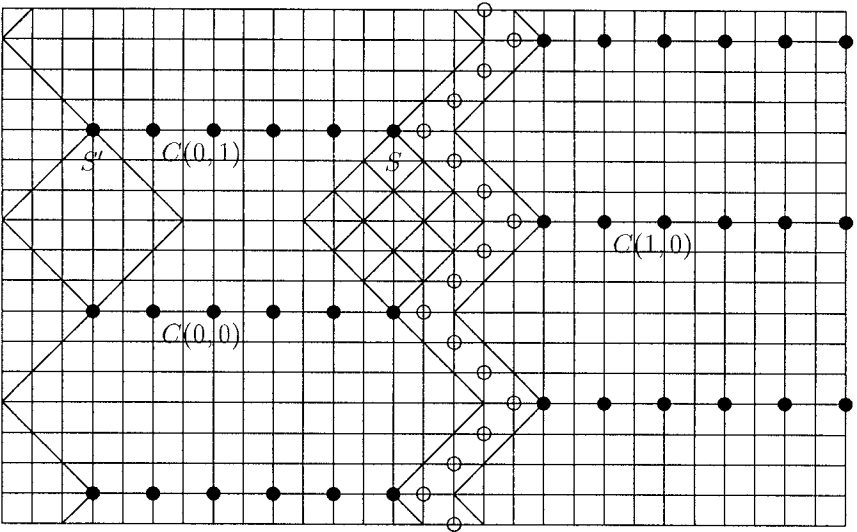


FIG. 3. The construction when $r = 6$. The codewords are indicated with solid dots.

All the remaining lattice points are included in one of an infinite number of infinitely long "corridors." The points of one such corridor are marked with open circles in Fig. 3. The "corner points" belong to one of the sets $L(i, j)$, and for any other point v the set $I(v)$ contains points only from $C(i, j)$ and $C(i+1, j)$ for some i and j , or alternatively only from $C(i, j)$ and $C(i+1, j-1)$. As with the points of $L(i, j)$ we can immediately identify v .

Without loss of generality, we can therefore assume that we already know that $v \in H(0, 0)$. If $I(v)$ contains both the end points $(2r-2, 0)$ and $(2r-2, r)$, then v is within a square, which we denote by S (again cf. Fig. 3). Similarly, let S' denote the square consisting of the points v in $H(0, 0)$ for which $I(v)$ contains both $(0, 0)$ and $(0, r)$.

If $v \in H(0, 0)$ is in neither S nor S' , then—by the definitions of S and S' — $I(v)$ has an empty intersection with $\{(0, 0), (2r-2, 0)\}$ or with $\{(0, r), (2r-2, r)\}$; say $(0, 0) \notin I(v)$ and $(2r-2, 0) \notin I(v)$. But then the pattern $I(v) \cap C(0, 0)$ is symmetric with respect to the line $x = v_x$, where v_x denotes the x -coordinate of v ; and so we immediately know v_x . Again we can identify v because if v moves down one step on the line $x = v_x$ (inside $H(0, 0)$), then we either lose (at least) one codeword of $C(0, 1)$ from $I(v)$ or gain (at least) one codeword of $C(0, 0)$.

Assume finally that $v \in S$. The case $v \in S'$ is of course symmetric. We can divide S into $r/2 \times r/2$ smaller squares as in Fig. 3; each such closed square contains five lattice points. Using only the points in $C(0, 0)$ and $C(0, 1)$ it is easy to find a little square which certainly contains v (there are usually more than one, and we take any one of them). Notice that all the lines (or rather, the line segments within S) that we are interested in are actually parts of boundaries of sets $B_r(c)$ for some $c \in C(0, 0)$ or $c \in C(0, 1)$. Given the little square where v lies, we can in fact check if v lies in the lower left-hand side (using a suitable codeword in $C(0, 0)$) and if v lies in the upper left-hand side (using a suitable codeword in $C(0, 1)$). Based on these two bits of information, we can identify v , unless v is the centre point or in the right-hand corner. But the right-hand side corner is always within distance r from one more codeword in $C(1, 0)$ than the centre point. ■

For odd r , we get asymptotically the same result.

THEOREM 3.2. *For every odd r , there is an r -identifying code with density $2r/(5r^2 - 2r + 1)$ in the square lattice.*

Proof. Let

$$C(0, 0) = \{(0, 0), (2, 0), (4, 0), \dots, (2r-2, 0)\}$$

and

$$C(i, j) = C(0, 0) + i \left(\frac{5r-3}{2}, \frac{r+1}{2} \right) + j(-1, r).$$

Then C , the union of all $C(i, j)$, $i, j \in \mathbb{Z}$, is an r -identifying code.

First we consider $M(0, 0)$, the set of all lattice points v for which $I(v)$ contains at least one codeword from each of the subcodes $C(0, 0)$, $C(0, 1)$ and $C(0, -1)$. This set consists of the points $(1, 0), (3, 0), \dots, (2r-3, 0)$, which clearly are identified by $C(0, 1)$ and $C(0, -1)$. The same holds for any set $M(i, j)$, $i, j \in \mathbb{Z}$, defined analogously.

Next we consider the codewords themselves. Any point at distance one from $C(0, 0)$ is at distance r from $C(0, 1)$ or $C(0, -1)$, and it is easy to see that $C(0, 0)$ is exactly the set of points v for which $I(v)$ consists of points in $C(0, 0)$ only. Moreover $C(0, 0)$ identifies the elements of $C(0, 0)$, and the same is true for any $C(i, j)$.

Now we consider $N(i, j)$, the set of all lattice points v other than the ones in $M(i, j)$ and $M(i, j+1)$ which are within distance r from at least one point in $C(i, j)$ and at least one point in $C(i, j+1)$. All points not in a set $C(i, j)$, $M(i, j)$ or $N(i, j)$ are included in one of an infinite number of “semi-corridors” and are marked with open circles in Fig. 4. These points are identified using the codewords in a set $C(k, \ell)$ together with $C(k+1, \ell)$.

Therefore our last case is when $v \in N(i, j)$, and without loss of generality, we can assume that $v \in N(0, 0)$. We define two rectangles S and S' as in the proof of the previous theorem ($v \in S$ if the codewords $(2r-2, 0)$ and

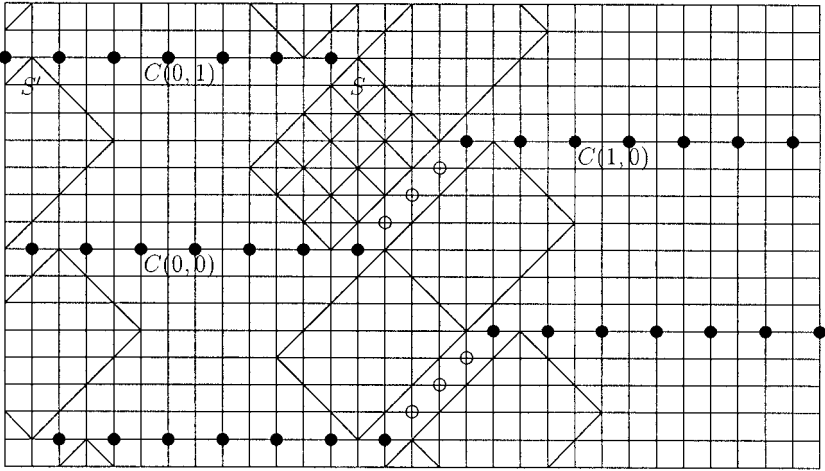


FIG. 4. The construction when $r = 7$. The codewords are indicated with solid dots.

$(2r-3, r)$ belong to $I(v)$, and the same argument applies when $v \notin S \cup S'$, because all distances from v to $C(0, 0)$ and $C(0, 1)$ have the same parity, hence moving on the vertical line $x = v_x$, we necessarily either gain or lose codewords.

So finally we can assume that $v \in S$. Again, S is divided into smaller squares of size five, and the key point is to observe that each centre of such a square is discriminated from its neighbour on the right by a codeword in $C(1, 0)$. ■

4. THE SPECIAL CASE $r = 2$

For small values of r it is natural that there are better constructions than the general ones described in the previous section. Fig. 5 shows a 2-identifying code with density $5/29$. The code has been constructed using a tile consisting of 29 vertices and 5 codewords (cf. Fig. 5).

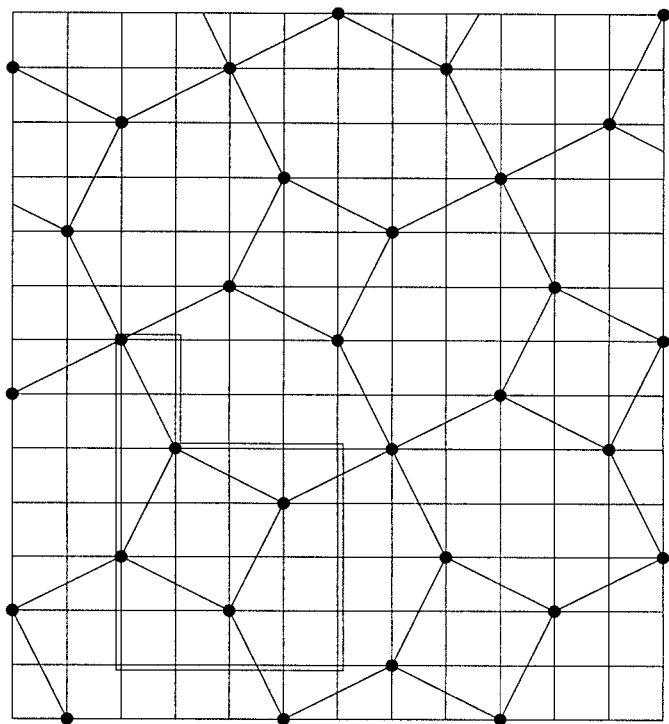


FIG. 5. A 2-identifying code with density $5/29$.

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